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1 Lecture 1

1.1 Signals

- Continuous-time $\Leftrightarrow$ analog $x(t)$, t is real
- Discrete-time $\Leftrightarrow$ digital $x(n)$, n is an integer

1.2 Samping

$$t \to nT_s$$

T_s: sampling periods
$F_s = \dfrac{1}{T_s}$: sampling frequency (Hz)

1.3 Shift

$$x(n)$$

- $x(n - n_0), n_0 > 0$ **right shift (delay)**
- $x(n + n_0), n_0 > 0$ **left shift (advance)**

1.4 Impulse

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

$$\delta(n - n_0) = \begin{cases} 1 & \text{if } n_0 = n \\ 0 & \text{if } n \neq n_0 \end{cases}$$

Any digital signal = sum of unit impulse with coefficients.

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n - k)$$

1.5 Unit Step

$$u(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

Unit step function expressed using impulse:

$$u(n) = \delta(n) + \delta(n - 1) + \delta(n - 2) + ...$$

Impulse function expressed using unit step function

$$\delta(n) = u(n) - u(n-1)$$

1.6 Exponential

$$\begin{aligned} A\beta^n &= |A|e^{j\angle\phi}(|\beta|e^{j\angle\theta})^n \\ &= |A||\beta|^n e^{j(\phi+n\theta)} \\ &= |A||\beta|^n(cos(\phi+n\theta)+jsin(\phi+n\theta)) \end{aligned}$$

1.7 Analog vs. Digital Frequency

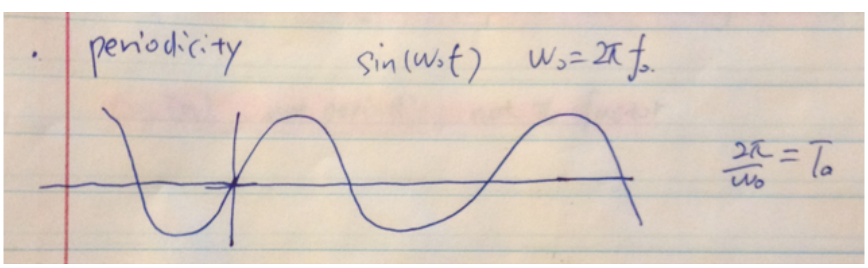
e.g.,

$$\begin{aligned} e^{j(\omega_0+2\pi k)n} &= e^{j\omega_0 n}e^{j2\pi kn} \\ &= e^{j\omega_0} \end{aligned}$$

ω_0 : Digital frequency
k : integer $\Rightarrow e^{j2\pi kn} = 1$
With digital frequency, you do not have to look the whole axis, because of a lot of redundancy. **Only need to look at a 2π interval, normally**

$$\begin{cases} -\pi \leq \omega_0 \leq \pi \\ 0 \leq \omega_0 < 2\pi \end{cases}$$

π is the highest possible frequency in discrete time



1.7.1 Digital Frequency

A continuous-time complex exponential can take any frequency in the range $-\infty$ *to* ∞, but when sampling it at rate F_s.

$$e^{j2\pi f_0 t} \rightarrow e^{j2\pi f_0 n T_s} \rightarrow e^{j2\pi \frac{f_0}{F_s} n} = e^{j\omega_0 n}$$

each time f_0 is a multiple of F_s, so that the ω_0 is an integer multiple of 2π (C-T frequency axis as being wrapped around a circle whose circumference is F_s Hz)

1.7.2 Units of ω_0

$$\omega_0 = \frac{2\pi f_0}{F_s} = \frac{(rad/cycle)(cycle/sec)}{samples/sec} = radians/sample$$

1.8 Discrete Time Signals

$x(n)$ is periodic if there exists an integer N(the period in D.T. has to be an integer):

$$x(n+N) = x(n)$$

e.g.,

$$\begin{aligned} cos(\omega_0 n) &= cos(\omega_0(n+N)) \\ &= cos(\omega_0 n + \omega_0 N) \end{aligned}$$

$$\omega_0 N = 2\pi k, \text{ k integer}$$

e.g., a discrete sinusoidal, $cos(\frac{\pi}{4}n)$,$\omega_0 = \frac{\pi}{4}$

$$\begin{aligned} \omega_0 N &= 2\pi k \\ N &= \frac{2\pi k}{\omega_0} = \frac{2\pi k}{\frac{\pi}{4}} \\ &= 8k \end{aligned}$$

N=8, $\boxed{\text{fundamental period}}$ for this discrete-time signal.

e.g., $cos(\frac{3\pi}{8}n)$, $\frac{3\pi}{8}$ is a higher frequency than $\frac{\pi}{4}$

$$N = \frac{2\pi k}{\frac{3\pi}{8}} = \frac{16k}{3} \Rightarrow N = 16, k = 3$$

$\Rightarrow$ **Period goes up as frequency increases. (whereas they are reciprocal in analog world)**

$cos(n)$**, not periodic; not π factor**

1.8.1 Summation (D.T.)

e.g., $x_1(n)$ has N_1 entries, and $x_2(n)$ has N_2 entries, their sum will always be periodic.

$$x_1(n) + x_2(n) = x_3(n)$$

$$\boxed{\text{always periodic}}$$

The period of the sum = gcd (N_1, N_2)

1.9 Systems

$$y(n) = T\{x(n)\}$$

$$\boxed{\text{y=Ax}}$$

1.10 Ideal Delay

$$x(n) \to x(n - n_0)$$

where $n_0 > 0$

1.11 Moving Average

$x(n) \to y(n) = \dfrac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x(n-k)$

1.12 Memoryless

$y(n)$ depends only on $x(n)$ at the same n.
e.g.,

$$y(n) = (x(n))^2$$

$$y(n) = 2x(n)$$

1.13 Linear System

$$T\{a_1 x_1(u) + a_2 x_2(u)\} = a_1 T\{x_1(u)\} + a_2 T\{x_2(u)\}$$

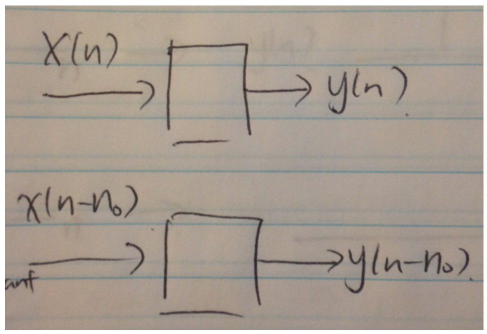
1.14 Accumulator

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)$$

2 Lecture 2

2.1 Time Invariant System

Shift input produces a corresponding shift in output.

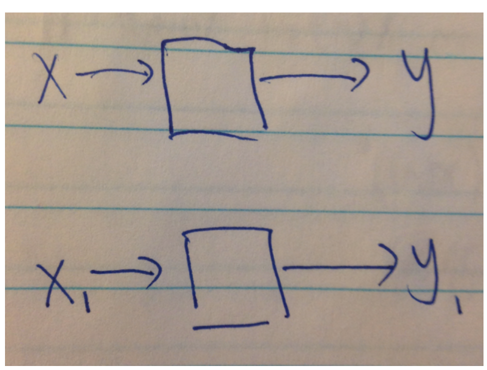


- $y(n-n_0)$- original output shifted
- $x(n-n_0)$- original input shifted

e.g., Time Invariant Systems

- Ideal delay
- Moving average
- Accumulator

e.g., given two systems shown below

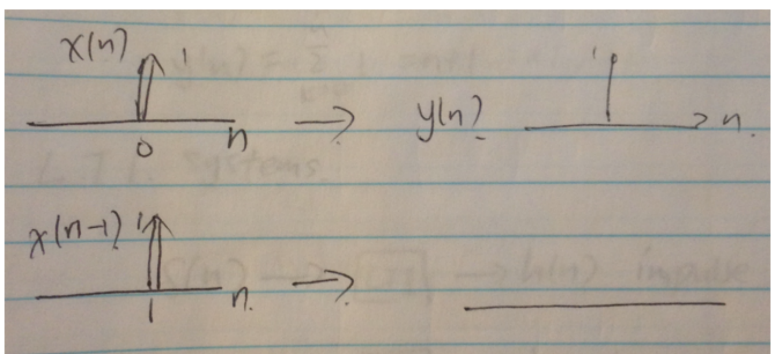


$$\begin{aligned}
y(n) &= \sum_{k=-\infty}^{n} x(k) \\
y(n-n_0) &= \sum_{k=-\infty}^{n-n_0} x(k) \\
y_1(n) &= \sum_{k=-\infty}^{n} x_1(k) = \sum_{k=-\infty}^{n} x(k-n_0) \\
&= \sum_{k_1=-\infty}^{n-n_0} x(k_1)
\end{aligned}$$

where $k_1 = k - n_0$. Note that $\sum_{k=-\infty}^{n-n_0} x(k)$ and $\sum_{k_1=-\infty}^{n-n_0} x(k_1)$ are the same $\Rightarrow$ $\boxed{\text{Time Invariant}}$.
e.g., Compressor (Not Time Invariant)

$$y(n) = x(Mn)$$

where M is a positive integer



2.2 Causal Systems

Output at $n = n_0$ depends only on input values from $n \leq n_0$

- Backward Difference

$$y(n) = x(n) - x(n-1)$$

which is **causal**

- Forward Difference

$$y(n) = x(n+1) - x(n)$$

which is **not causal**

2.3 Stable Systems

Bounded Input Bounded Output (BIBO)

$$|x(n)| \leq B < \infty$$

e.g., Accumulator, $y(n) = \sum_{k=-\infty}^{n} x(k)$
Take $x(n) = Unit(n)$, where

$$Unit(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

Now, $y(n) = \sum_{k=0}^{n} 1 = n + 1$

2.4 L.T.I. Systems

- Impulse Response
$$\delta(n) \to \boxed{\text{LTI}} \to h(n)$$
- T.I.
$$\delta(n-k) \to \boxed{\text{LTI}} \to h(n-k)$$
- Linear
$$x(k)\delta(n-k) \to \boxed{\text{SYS}} \to x(k)h(n-k)$$
$$\sum x(k)\delta(n-k) \to \boxed{\text{SYS}} \to \sum x(k)h(n-k) = y(n)$$
- In fact, $y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$ is the convolution between $x(n)$ and $h(n)$, a flip and shift to get $h(n-k)$

2.4.1 Convolution Commutes

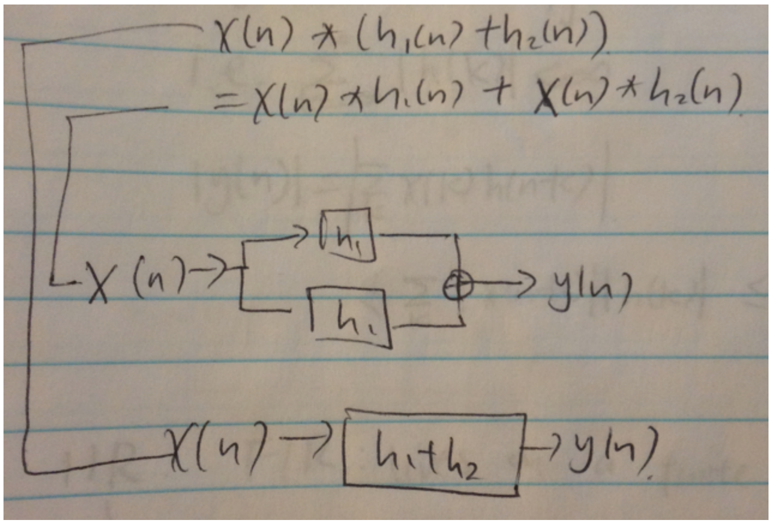
$$x(n) * h(n) = h(n) * x(n)$$

$$x(n) \to \boxed{\text{h(n)}} \to y(n)$$

$$h(n) \to \boxed{\text{x(n)}} \to y(n)$$

2.4.2 Convolution distributes over addition

$$x(n) * (h_1(n) + h_2(n)) = x(n) * h_1(n) + x(n) * h_2(n)$$



2.4.3 Associative

$$\begin{aligned} x(n) * h_1(n) * h_2(n) &= (x(n) * h_1(n)) * h_2(n) \\ &= (x(n) * h_2(n)) * h_1(n) \\ &= x(n) * (h_1 * h_2) \end{aligned}$$

2.5 L.T.I case

A system is stable iff $h(n)$ is absolutely summable.
i.e.,
$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$
$|y(n)| = |\sum_k x(k)h(n-k)| \leq \sum_k |x(n-k)||h(k)| \leq B_x \sum_k |h(k)|$

2.6 FIR(Finite Impulse Response) and IIR

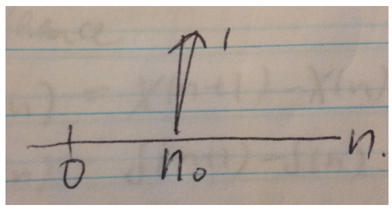
IIR stands for infinite length impulse response(not desirable).
FIR lives on a finite interval on an axis.
A L.T.I. system is causal, iff $h(n) = 0$ for $n < 0$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

where $n-k<n$, therefore k can't be negative.

2.6.1 Ideal delay

$y(n) = x(n-n_0)$, $n_0 > 0$, replace x with δ
$h(n) = \delta(n-n_0)$



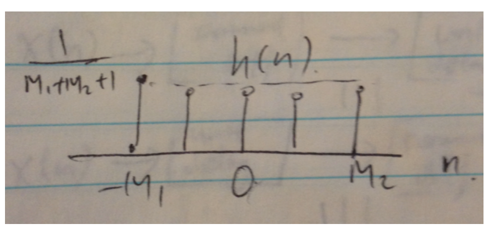
Conclusion: A causal and stable system.

2.7 Moving Average Filter

The moving average filter is not causal, because it's got stuff on the left of origin.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$h(n) = \frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M_2}\delta(n-k)\begin{cases}\frac{1}{M_1+M_2+1} & \text{if } -M_1 \le n \le M_2\\ 0 & \text{if } otherwise\end{cases}$$



2.8 Accumulator

The impulse response of an accumulator is a unit step function.

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

$$h(n) = \sum_{k=-\infty}^{n} \delta(k) = U(n)$$

which always gives 1 when $k = 0$. Therefore,

$$h(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

2.9 Forward difference and Backward difference

- Forward difference, **FIR-Stable**

$$y(n) = x(n+1) - x(n)$$

$$h(n) = \delta(n+1) - \delta(n)$$

- Backward difference, **FIR-Stable and Causal**

$$y(n) = x(n) - x(n-1)$$

$$h(n) = \delta(n) - \delta(n-1)$$

3 Lecture 3

3.1 IIR-Stable

e.g., $h(n) = a^n u(n)$, where a^n is a decaying exponential($|a| < 1$), a can be real or complex, as long as its magnitude is less than 1.

$$\sum_{n=0}^{\infty} |a|^n = \frac{1}{1 - |a|}$$

3.2 FIR

$$x(n) \rightarrow \boxed{\text{Forward difference}} \rightarrow \boxed{\text{Unit delay}} \rightarrow y(n)$$

$$x(n) \rightarrow \boxed{\text{Unit delay}} \rightarrow \boxed{\text{Forward difference}} \rightarrow y(n)$$

$$x(n) \rightarrow \boxed{\text{Backward difference}} \rightarrow y(n)$$

- Forward difference [not causal]

$$h(n) = \delta(n+1) - \delta(n)$$

- Backward difference [Forward * Unit delay; causal]

$$h(n) = \delta(n) - \delta(n-1)$$

- Unit delay

$$h(n) = \delta(n-1)$$

For example:

$$\begin{aligned}[\delta(n+1) - \delta(n)] * \delta(n-1) &= \delta(n+1) * \delta(n-1) - \delta(n) * \delta(n-1) \\ &= \delta(n) - \delta(n-1)\end{aligned}$$

Note: $\delta(n-n_0)$ is like a **shift operator**, which acts as follows: $g(n) * \delta(n - n_0) = g(n-n_0)$. Therefore, the first term is shifted to the right one unit and is equal to $\delta(n)$.
Any FIR systems can be made causal by cascading with a delay

3.3 Inverse Systems

Definition:

$$x(n) \rightarrow \boxed{\text{SYS}} \rightarrow y(n)$$

$$y(n) \rightarrow \boxed{\text{Inv SYS}} \rightarrow x(n)$$

e.g.,

$$x(n) \rightarrow \boxed{\text{Accumulator}} \rightarrow \boxed{\text{Backward difference}} \rightarrow x(n)$$

Note: The backward difference filter acts like an inverse system to the accumulator applied before it. We can express the above relation mathematically as follows:

$$\begin{aligned}\delta(n) &= U(n) * [\delta(n) - \delta(n-1)] \\ &= U(n)\delta(n) - U(n)\delta(n-1) \\ &= U(n) - U(n-1) \\ &= \delta(n)\end{aligned}$$

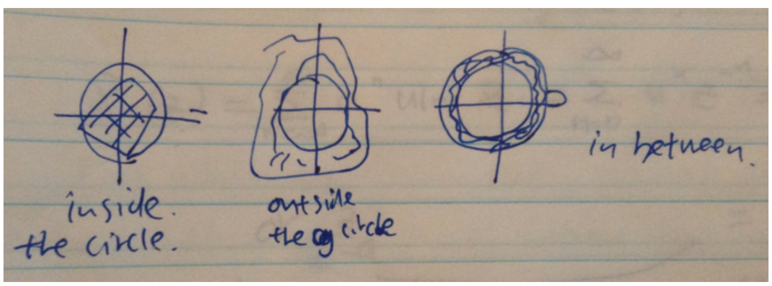
3.4 Z-transform (Digital version of Laplace Transform)

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

A bilateral transform; z is complex.
Set of z values for which $|X(z)| < \infty$ is called the **region of convergence(ROC)**. $\Rightarrow$ the Sum is finite.

$$|X(z)| = \sum_{n=-\infty}^{\infty} |x(n)||z|^{-n}$$

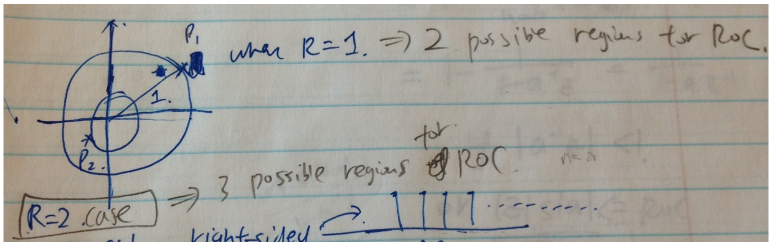
Convergence depends only on $|z|$ not $\angle z$
ROC is a disk in a complex plane.



$$X(z) = \frac{P(z)}{Q(z)}$$

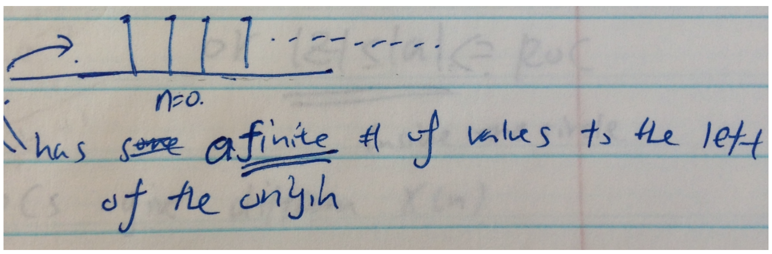
Assume no common factor in P,Q
Values of z for which $X(z) = 0$ are called "zeros".
Values of z for which $X(z) = \infty$ are called "poles".

- If there are N poles with R distinct magnitudes.($N \leq R$, multiple poles can have same magnitude)
- ROC can't contain any poles.
- $x(n) \rightarrow X(z)$ **unique**
- $X(z) \rightarrow x(n)$ **not unique** (needs to know ROC to get back to the time domain)



3.5 Causal,right-sided, $|z| > |a|$

$$x(n) = a^n u(n)$$

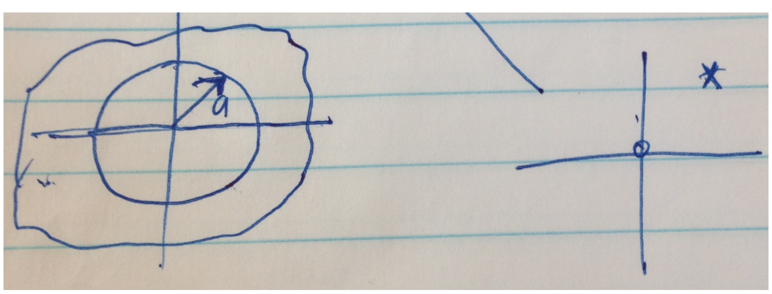


$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}$$

Not to blow up $\sum_{n=0}^{\infty}(az^{-1})^n$, we need $|az^{-1}| < 1$ or $|z| > |a|$, which is the ROC.
or

$$\frac{z}{z-a}$$

Pole at $z = a$, zero at $z = 0$



3.6 Left-sided,$|z| < |a|$

$x(n) = -a^n u(-n-1) \rightarrow x(n) = 0$ if $n \geq 0$, as long as $n \geq 0$, get zero.

$$\begin{aligned}
X(z) &= -\sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n} \\
&= -\sum_{n=-\infty}^{-1} a^n z^{-n} \\
&= -\sum_{n=1}^{\infty} a^{-n} z^n + (-a^{-0} z^0) \\
&= 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n \\
&= 1 - \frac{1}{1 - a^{-1} z} = \frac{1}{1 - a z^{-1}}
\end{aligned}$$

$|a^{-1} z| < 1$ or $|z| < |a|$ is its ROC.
Conclusion

- different ROCs give different $x(n)$
- right-sided sequence - ROC outside circle
- left-sided sequence - ROC inside circle
- Two-sided sequence - ROC between two circles

e.g., a 2-pole example. Given that $x(n) = (\frac{1}{2})^n u(n) + (\frac{-1}{3})^n u(n)$, its z-transform is

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}}$$

we know the ROCs for the two terms are $|z| > \frac{1}{2}$ and $|z| > \frac{1}{3}$
If we proceed, after the multiplication we will get the following expression

$$X(z) = \frac{2(1 - \frac{1}{12} z^{-1})}{(1 - \frac{1}{2} z^{-1})(1 + \frac{1}{3} z^{-1})}$$

We want to take the intersection of the two ROCs, which gives us

$$|z| > \frac{1}{2}$$

Poles: $z = \frac{1}{2}, -\frac{1}{3}$
Zeros: $z = \frac{1}{12}, 0$
e.g., Now $x(n) = (-\frac{1}{3})^n u(n) - (\frac{1}{2})^n u(-n-1)$, its z-transform is

$$X(z) = \frac{1}{1 + \frac{1}{3}z^{-1}} + \frac{1}{1 - \frac{1}{2}z^{-1}}$$

Note that the first term in $x(n)$ is right-sided ($|z| > |a|$) and the second term is left-sided ($|z| < |a|$).
Therefore, the ROC for the first term is $|z| > \frac{1}{3}$ and is $|z| < \frac{1}{2}$ for the second term.
Overall ROC: $\frac{1}{3} < |z| < \frac{1}{2}$

- If $x(n)$ is right-sided, ROC is region outside. Also, outermost finite pole to include possibly $z = \infty$. Also note that, if $n < 0$, $X(z) = \sum x(n)z^{-n}$ will blow up. So to include $z = \infty$, we need $n > 0$
- If $x(n)$ is left-sided, ROC is region inside innermost non zero-pole to possibly include $z = 0$

3.7 Partial Fraction Expansion

$$X(z) = \alpha \frac{\Pi_{k=1}^{M}(1 - c_k z^{-1})}{\Pi_{k=1}^{N}(1 - d_k z^{-1})}$$

Assume $M < N$ and all poles are first order.
$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$ and $A_k = (1 - d_k z^{-1})X(z)|_{z=d_k}$
e.g.,

$$X(z) = \frac{1}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})} = \frac{A_1}{1 - \frac{1}{4}z^{-1}} + \frac{A_2}{1 - \frac{1}{2}z^{-1}}$$

$$A_1 = (1 - \frac{1}{4}z^{-1})X(z)|_{z=\frac{1}{4}} = \frac{1}{1 - \frac{1}{2}z^{-1}}|_{z=\frac{1}{4}} = -1$$

$$A_2 = (1-\frac{1}{2}z^{-1})X(z)|_{z=\frac{1}{2}} = \frac{1}{1-\frac{1}{4}z^{-1}}|_{z=\frac{1}{2}} = 2$$

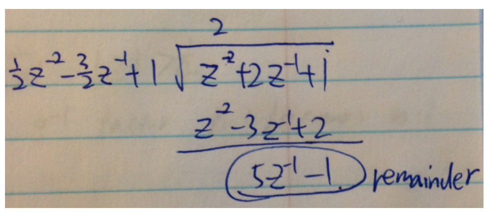
Therefore,

$$x(n) = -(\frac{1}{4})u(n) + 2(\frac{1}{2})^n u(n)$$

e.g., $M > N$, do long division.

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}$$

$B_r z^{-r} \to B_r \delta(n-r)$
ROC: $|z| > 0$
e.g., $X(z) = \dfrac{(1+z^{-1})^2}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$, do a long division:



We have

$$2 + \frac{-1+5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} = 2 - \frac{9}{1-\frac{1}{2}z^{-1}} - \frac{8}{1-z^{-1}}$$

Hence,

$$x(n) = 2\delta(n) - 9(\frac{1}{2})^n u(n) + 8u(n)$$

e.g.,we know $X(z) = \sum x(n)z^{-n}$. Given $X(z) = z^2(1-\frac{1}{2}z^{-1})(1+z^{-1})(1-z^{-1}) = Z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$. We can directly write out

$$x(n) = \delta(n+2) - \frac{1}{2}\delta(n+1) - \delta(n) + \frac{1}{2}\delta(n-1)$$